THURSDAY, SEPTEMBER 9, 1897.

ABELIAN AND THETA FUNCTIONS.

Abel's Theorem and the allied Theory, including the Theory of the Theta Functions. By H. F. Baker, M.A. Pp. xx + 684. (Cambridge: at the University Press, 1897.)

CAYLEY'S often-quoted simile which compares the province of mathematics with "a tract of beautiful country seen at first in the distance, but which will bear to be rambled through and studied in every detail of hillside and valley, stream, rock, wood, and flower," suggests a comparison of mathematical text-books with those useful works which provide information and advice for the tourist and the traveller. It may be said with truth that there is every variety, from the cheap illustrated pamphlet, designed to catch the eye of the holiday tripper in search of the picturesque, to the elaborate maps, surveys, and gazetteers which are best appreciated by the genuine explorer.

It is to the latter class that Mr. Baker's treatise belongs. It is in no sense a book for beginners; in order to appreciate it, the reader must be tolerably familiar with modern function-theory, and not wholly ignorant of the special subject of the work. Even then, he will probably find the treatise most useful as a "hand-book"; that is to say, as a methodical guide to, and commentary upon, the host of memoirs which are ultimately connected with Abel's researches on algebraic integrals. In this respect, the work is sure to be very valuable; for the author has evidently spared no pains in making himself familiar with everything of importance that has been done in this field, and in attaining, so far as possible, an impartial attitude towards the different ways in which the subject has been presented.

The title recalls the fact that not so very long ago "Abel's Theorem" was apt to be regarded by mathematical students as being something like "Taylor's Theorem" or "Ivory's Theorem": a stiff piece of isolated analysis, for which one's "coach" was expected to produce a concise and elegant proof, adapted for writing out in an examination. Even now it may be a surprise to some readers to find that a work professing to deal mainly with Abel's Theorem can extend to nearly seven hundred large and well-filled pages of print. The truth is that while Abel's Theorem is in itself a comparatively simple matter, and is perhaps best regarded as a theorem in symmetric functions, it forms a kind of centre from which several fundamental theories may be said to radiate. In order to give precision to the theorem itself, it is necessary to establish the real nature and properties of algebraic functions; Jacobi's problem of inversion, or, in other words, the introduction of Abelian functions properly so called, leads to the vast and in some ways still mysterious theory of the general theta-functions; these, in their turn, have in late years suggested an almost bewildering variety of transcendental functions; and, finally, we have the reaction of all these discoveries upon analytical geometry. It is therefore not astonishing that Mr. Baker's volume is so large; it would, in fact, have been much larger but for his studied conciseness, which, indeed, here and there, may be thought almost overdone.

In a subject like this, which, perhaps more than any other, has brought out the contrast between the intuitional and analytical schools, it is always interesting to observe an author's point of view. Mr. Baker has adopted a kind of middle course which will commend itself to those whose minds, like Cayley's, while essentially analytical, love to clothe their discoveries in the language of geometry. Thus the real start is made by assuming the existence of Riemann's fundamental integrals of the first, second, and third kind. Of course, after the work of Neumann, Schwarz, and others, there is no logical inconsistency in this; and it is convenient for purposes of exposition. But it is open to two objections: the first is, that the proof of Riemann's existence-theorems is long and difficult, when it is thorough; the second, and more essential, is that algebraic functions present themselves as special collocations of integrals, and are not considered in the first instance on their own merits; it is hard to refrain from thinking that this is, to some extent, putting the cart before the horse.

But it ought in fairness to be said, that in subsequent chapters (iii.–vii.), the analytical theory of Weierstrass, Dedekind, Kronecker, and Hensel is explained in sufficient detail to enable the reader to appreciate the other point of view; although, as a matter of fact, Weierstrass's fundamental "Lückensatz" is deduced from the existence-theorems.

Points which deserve attention in these earlier chapters are the careful explanation of "the places and infinitesimal on a Riemann surface"; the discussion of the Riemann-Roch theorem, which is unusually clear, and well illustrated by examples; and the satisfactory treatment of adjoint polynomials. As to this last point, we cannot but think that the advantage is all on the side of the analytical school. They can explain why, and in what sense, a singular point on a curve is to be reckoned as δ nodes and κ cusps in the calculation of the deficiency; can the intuitional mathematician say what δ and κ ought to be at a higher singularity of a curve described by a definite mechanical process, without finding the equation of the curve?

Chapter viii. dismisses Abel's Theorem in the course of twenty-seven pages. This brief treatment is made possible by what has gone before; still it is rather a pity that more space has not been given to an independent and purely algebraical treatment of at least the differential form of the theorem.

In chapter ix. we are introduced to the inversion problem: and here it is pleasing to find an account of Weierstrass's procedure, which is certainly the most satisfactory in giving an intelligible form to the result, although, of course, it is not a practicable method of solving the problem. For this we must have recourse to the theta-functions, and these are in fact introduced in chapter x. It is with the theta-functions and their properties that the rest of the book is principally concerned; chapters x., xi., xv.–xxi. are almost entirely devoted to them, and give, in fact, the most elaborate systematic treatment of the functions that has yet appeared

U

© 1897 Nature Publishing Group

in English. The conciseness of the notation and the difficulty of the subject make these chapters very hard reading. Every one acquainted with the subject is aware that the multiple theta-functions sprang from a brilliant generalisation of Jacobi's functions $\Theta(u)$, $H(u)$, and that the *crux* of the whole matter is to establish, in a natural way, their connection with a set of algebraical functions. It is impossible to feel that this connection has been made in the really proper and natural way: the theta-functions drop out of the clouds, so to speak, and are joined up to the inversion problem in a more or less artificial and unsatisfactory manner. The most promising glimpse of a better method seems to be afforded by the "Prime Form" introduced by Schottky and Klein; and it is very satisfactory that due prominence is given to this function in chapter xiv. (on factorial functions) and elsewhere. It is very important to notice that this function admits of an independent definition, and that the three kinds of elementary Riemann functions may be directly and simply derived from it (see Arts. 233-5). Moreover there is a relation connecting any theta-function with a corresponding prime form (Arts. 237, 272-4). Thus the prime form has a strong claim to be considered fundamental; and it is not improbable that, by starting with it, the same kind of simplification may be attained as that which has been achieved in the theory of elliptic functions by the use of Weierstrass's sigma-function.

It would be unprofitable to attempt to give an abstract of the chapters on the theta-functions which Mr. Baker's book contains; enough to say that (besides the elementary properties) the relations among theta-products, the theory of transformation, and the theory of characteristics are all fully treated; account being taken not only of memoirs already classical, such as those of Rosenhain, Göpel, Hermite, and Weierstrass, but also of quite recent researches, such as those of Prym, Frobenius, Poincaré, and others. A certain amount of special attention is given to the hyper-elliptic case; this is justified by its comparative simplicity, and the large amount of literature connected with it.

Some parts of the book, especially the last two chapters (on complex multiplication of theta-functions, the theory of correspondences, and degenerate Abelian integrals), deal with problems of which the complete solution is still the object of research. This is a very welcome feature; for the unavoidable incompleteness of these parts is more than compensated by their stimulating quality. It is a pity that authors of mathematical treatises too often neglect the opportunity of carrying on a discussion to actual contact with current research, and pointing out the possible or probable direction of future development.

One characteristic of Mr. Baker's treatise seems to call for remark. On p. 93 the author says: "We desire to specify all the possibilities"; this sentence might have been adopted as a motto for the whole work. There is such a wealth of conscientious detail that we can imagine some readers failing to grasp the general argument, and becoming disheartened by the array of complicated formulæ with their plentiful adornment of suffixes. Of course, in order to treat the subject generally, a certain amount of complexity is unavoidable; there is, however, an alternative which can often be adopted, and is worth considering, namely, instead of introducing n symbols, say $x_1, x_2 \dots x_n$, to use a limited number, say $x_1 x_2 x_3$ or $x y z$, and to give the demonstration in such a form that its generality can be inferred. It must be admitted that this course is not always practicable; the fact is that, in order to read modern analysis with comfort, a certain facility in handling sums and products in a condensed notation is almost indispensable, and should be acquired, if possible, at an early age, as in the analogous case of definite integrals.

The book contains a considerable number of illustrative examples, many of which are worked out in detail. These cannot but be of great help to the reader, by showing how the general theory is brought to bear upon particular cases. This is especially true in the actual construction of the Riemann integrals for a given plane curve.

Printers' errors appear to be very rare; on p. 138, line 14 from the bottom, "not greater than $Q - p$" should be "not less than $Q - p$"; and in the early part of Art. 176 there are several misprints, which, however, the reader can easily correct for himself.

There can be no doubt that this work will be highly appreciated by all who make a special study of Abelian functions; and we trust that, in the approbation of the limited circle to which he appeals, Mr. Baker will find a sufficient reward for the immense amount of labour which his task must have entailed. There is still room for a strictly introductory work, bringing out the salient features of the theory, and perhaps not disdaining "heuristic" methods of investigation. In spite of its limited scope and occasional diffuseness, there is a charm about C. Neumann's book which we miss in the more analytical treatises; something of this kind in English would probably do much to draw attention to this very fascinating field of research, and induce a select few to follow up the somewhat abstruse analysis which a more detailed study of the subject involves. Some reference to the historical evolution of the theory would not be out of place; indeed, we rather regret that the plan of Mr. Baker's treatise has tended to obscure this side of the matter. Cauchy is only mentioned once, and Puiseux not at all; yet the work of these two mathematicians was fundamental, and will always form a part of any systematic discussion of function-theory.

G. B. M.

THE CULTURE OF FRUIT.

The Principles of Fruit-Growing. By L. H. Bailey. Pp. xi + 508. (New York: The Macmillan Company. London: Macmillan and Co., Ltd., 1897.)

FRUIT-GROWING in this country is one of the remedies proposed to counterbalance the effects of low prices for agricultural products. But fruit-growing is an art which cannot be learned without experience. It is no easy matter for a farmer to change his habits and his practices, even if the local conditions are favourable to the production of fruit; and the orchard of the farm is very generally the most neglected part of the whole establishment. Nevertheless, it is obvious that a great extension of fruit culture has taken place during

© 1897 Nature Publishing Group